OCR
Oxford Cambridge and RSA

Tuesday 20 June 2023 – Afternoon

A Level Mathematics A

H240/03 Pure Mathematics and Mechanics

Time allowed: 2 hours

You must have:
- the Printed Answer Booklet
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to **3** significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

Formulae
A Level Mathematics A (H240)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$S_n = \frac{a(1-r^n)}{1-r}$

$S_\infty = \frac{a}{1-r}$ for $|r| < 1$

Binomial series

$(a+b)^n = a^n + {}^n\mathrm{C}_1 a^{n-1}b + {}^n\mathrm{C}_2 a^{n-2}b^2 + \ldots + {}^n\mathrm{C}_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$

where ${}^n\mathrm{C}_r = {}_n\mathrm{C}_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$

Quotient rule $y = \frac{u}{v}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$

Integration

$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$

$\int \mathrm{f}'(x)(\mathrm{f}(x))^n\,\mathrm{d}x = \frac{1}{n+1}(\mathrm{f}(x))^{n+1} + c$

Integration by parts $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Standard deviation

$\sqrt{\dfrac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\dfrac{\Sigma x^2}{n} - \bar{x}^2}$ **or** $\sqrt{\dfrac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\dfrac{\Sigma fx^2}{\Sigma f} - \bar{x}^2}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, mean of X is np, variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $\mathrm{P}(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Kinematics

Motion in a straight line	Motion in two dimensions
$v = u + at$	$\mathbf{v} = \mathbf{u} + \mathbf{a}t$
$s = ut + \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$
$s = \frac{1}{2}(u + v)t$	$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$
$v^2 = u^2 + 2as$	
$s = vt - \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Section A
Pure Mathematics

1 Using logarithms, solve the equation

$4^{2x+1} = 5^x$,

giving your answer correct to **3** significant figures. **[3]**

2 **(a)** Express $3\sin x - 4\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to **4** significant figures. **[3]**

(b) Hence solve the equation $3\sin x - 4\cos x = 2$ for $0° < x < 90°$, giving your answer correct to **3** significant figures. **[2]**

3 The cubic polynomial $f(x)$ is defined by $f(x) = x^3 + px + q$, where p and q are constants.

(a) **(i)** Given that $f'(2) = 13$, find the value of p. **[2]**

(ii) Given also that $(x-2)$ is a factor of $f(x)$, find the value of q. **[2]**

The curve $y = f(x)$ is translated by the vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

(b) Using the values from part **(a)**, determine the equation of the curve after it has been translated. Give your answer in the form $y = x^3 + ax^2 + bx + c$, where a, b and c are integers to be found. **[4]**

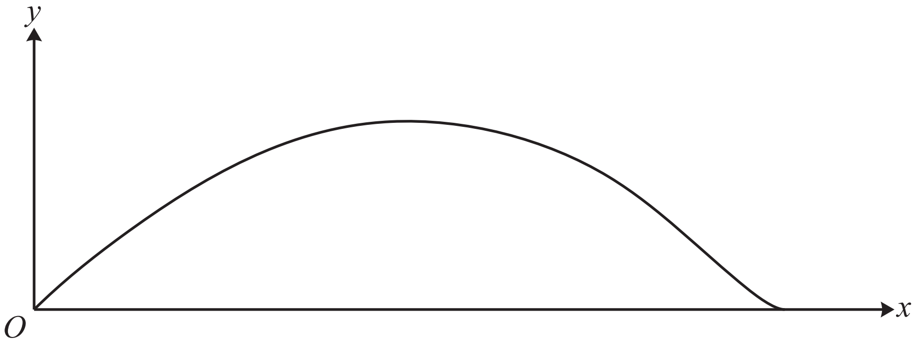
4 A circle C has equation $x^2 + y^2 - 6x + 10y + k = 0$.

(a) Find the set of possible values of k. **[2]**

(b) It is given that $k = -46$.

Determine the coordinates of the **two** points on C at which the gradient of the tangent is $\frac{1}{2}$. **[5]**

5 A mathematics department is designing a new emblem to place on the walls outside its classrooms. The design for the emblem is shown in the diagram below.



The emblem is modelled by the region between the x-axis and the curve with parametric equations

$x = 1 + 0.2t - \cos t, \qquad y = k\sin^2 t,$

where k is a positive constant and $0 \leqslant t \leqslant \pi$.

Lengths are in metres and the area of the emblem must be $1\,\text{m}^2$.

(a) Show that $k\int_0^{\pi}(0.2 + \sin t - 0.2\cos^2 t - \sin t\cos^2 t)\,dt = 1$. **[3]**

(b) Determine the exact value of k. **[6]**

6 The first, third and fourth terms of an arithmetic progression are u_1, u_3 and u_4 respectively, where

$u_1 = 2\sin\theta, \qquad u_3 = -\sqrt{3}\cos\theta, \qquad u_4 = \frac{7}{2}\sin\theta,$

and $\frac{1}{2}\pi < \theta < \pi$.

(a) Determine the exact value of θ. **[3]**

(b) Hence determine the value of $\sum_{r=1}^{100} u_r$. **[3]**

7 A car C is moving horizontally in a straight line with velocity $v\,\mathrm{m\,s^{-1}}$ at time t seconds, where $v > 0$ and $t \geqslant 0$. The acceleration, $a\,\mathrm{m\,s^{-2}}$, of C is modelled by the equation

$$a = v\left(\frac{8t}{7+4t^2} - \frac{1}{2}\right).$$

(a) **In this question you must show detailed reasoning.**

Find the times when the acceleration of C is zero. **[3]**

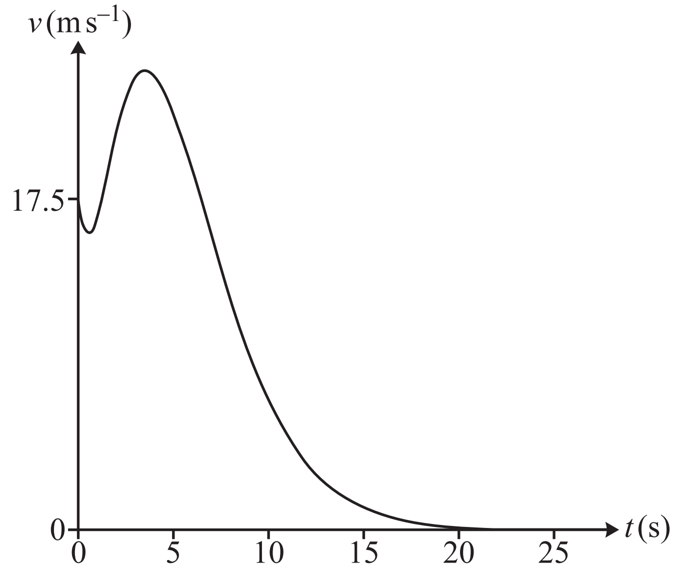
At $t = 0$ the velocity of C is $17.5\,\mathrm{m\,s^{-1}}$ and at $t = T$ the velocity of C is $5\,\mathrm{m\,s^{-1}}$.

(b) By setting up and solving a differential equation, show that T satisfies the equation

$$T = 2\ln\left(\frac{7+4T^2}{2}\right).$$ **[6]**

(c) Use an iterative formula, based on the equation in part **(b)**, to find the value of T, giving your answer correct to **4** significant figures. Use an initial value of 11.25 and show the result of each step of the iteration process. **[2]**

(d) The diagram below shows the velocity-time graph for the motion of C.



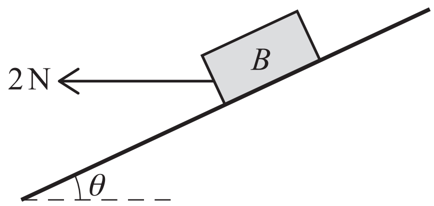
Find the time taken for C to decelerate from travelling at its maximum speed until it is travelling at $5\,\mathrm{m\,s^{-1}}$. **[1]**

Section B
Mechanics

8 A particle P moves with constant acceleration $(3\mathbf{i}-2\mathbf{j})\,\mathrm{m\,s^{-2}}$. At time $t=4$ seconds, P has velocity $6\mathbf{i}\,\mathrm{m\,s^{-1}}$.

Determine the speed of P at time $t=0$ seconds. **[4]**

9



A block B of weight 10 N lies at rest in equilibrium on a rough plane inclined at θ to the horizontal. A horizontal force of magnitude 2 N, acting above a line of greatest slope, is applied to B (see diagram).

(a) Complete the diagram in the Printed Answer Booklet to show all the forces acting on B. **[1]**

It is given that B remains at rest and the coefficient of friction between B and the plane is 0.8.

(b) Determine the greatest possible value of $\tan\theta$. **[5]**

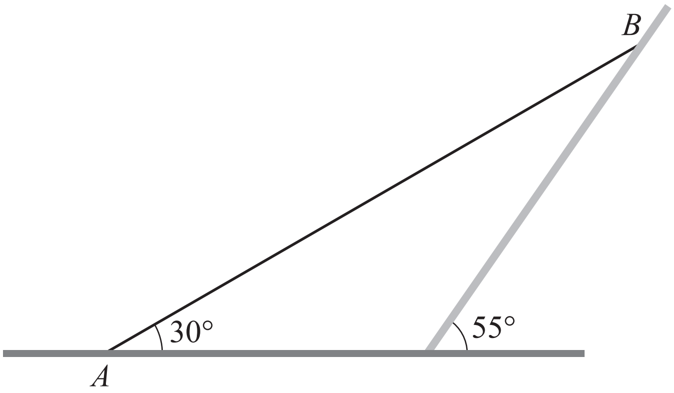
10 A particle P of mass m kg is moving on a smooth horizontal surface under the action of two constant horizontal forces $(-4\mathbf{i}+2\mathbf{j})\,\mathrm{N}$ and $(a\mathbf{i}+b\mathbf{j})\,\mathrm{N}$. The resultant of these two forces is $\mathbf{R}$ N. It is given that $\mathbf{R}$ acts in a direction which is parallel to the vector $-\mathbf{i}+3\mathbf{j}$.

(a) Show that $3a+b=10$. **[3]**

It is given that $a=6$ and that P moves with an acceleration of magnitude $5\sqrt{10}\,\mathrm{m\,s^{-2}}$.

(b) Determine the value of m. **[4]**

11

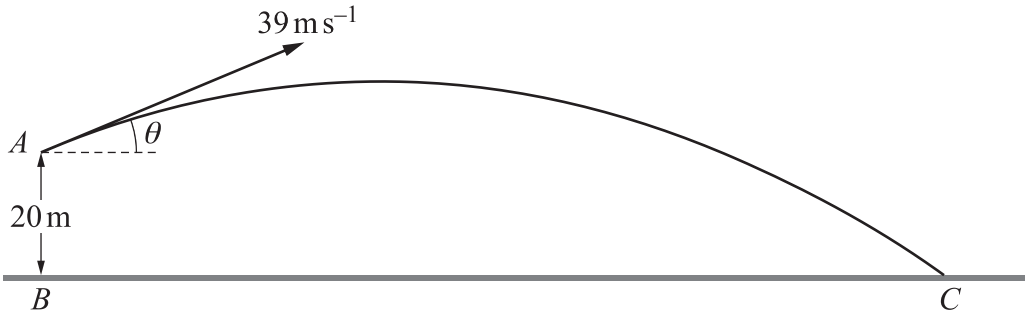


A uniform rod AB, of weight 20 N and length 2.8 m, rests in equilibrium with the end A in contact with rough horizontal ground and the end B resting against a smooth wall inclined at 55° to the horizontal. The rod, which rests in a vertical plane that is perpendicular to the wall, is inclined at 30° to the horizontal (see diagram).

(a) Show that the magnitude of the force acting on the rod at B is 9.56 N, correct to **3** significant figures. **[3]**

(b) Determine the magnitude of the contact force between the rod and the ground. Give your answer correct to **3** significant figures. **[5]**

12 In this question you should take the acceleration due to gravity to be $10\,\mathrm{m\,s^{-2}}$.



A small ball P is projected from a point A with speed $39\,\mathrm{m\,s^{-1}}$ at an angle of elevation θ, where $\sin\theta = \frac{5}{13}$ and $\cos\theta = \frac{12}{13}$. Point A is 20 m vertically above a point B on horizontal ground. The ball first lands at a point C on the horizontal ground (see diagram).

The ball P is modelled as a particle moving freely under gravity.

(a) Find the maximum height of P above the ground during its motion. **[3]**

The time taken for P to travel from A to C is T seconds.

(b) Determine the value of T. **[3]**

(c) State **one** limitation of the model, other than air resistance or the wind, that could affect the answer to part **(b)**. **[1]**

At the instant that P is projected, a second small ball Q is released from rest at B and moves towards C along the horizontal ground.

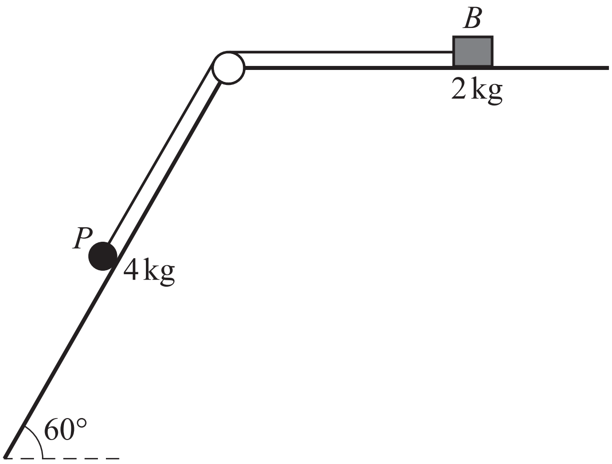
At time t seconds, where $t \geqslant 0$, the velocity $v\,\mathrm{m\,s^{-1}}$ of Q is given by

$$v = kt^3 + 6t^2 + \frac{3}{2}t,$$

where k is a positive constant.

(d) Given that P and Q collide at C, determine the acceleration of Q immediately before this collision. **[6]**

13



The diagram shows a small block *B*, of mass 2 kg, and a particle *P*, of mass 4 kg, which are attached to the ends of a light inextensible string. The string is taut and passes over a small smooth pulley fixed at the intersection of a horizontal surface and an inclined plane. The particle can move on the inclined plane, which is rough, and which makes an angle of 60° with the horizontal. The block can move on the horizontal surface, which is also rough.

The system is released from rest, and in the subsequent motion *P* moves down the plane and *B* does not reach the pulley.

It is given that the coefficient of friction between *P* and the inclined plane is twice the coefficient of friction between *B* and the horizontal surface.

(a) Determine, in terms of *g*, the tension in the string. **[7]**

When *P* is moving at $2\,\mathrm{m\,s^{-1}}$ the string breaks. In the 0.5 seconds after the string breaks *P* moves 1.9 m down the plane.

(b) Determine the deceleration of *B* after the string breaks. Give your answer correct to **3** significant figures. **[5]**

END OF QUESTION PAPER

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